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WHY SHOULD WE SPEAK ABOUT A COMPLEMENTARITY OF SENSE AND REFERENCE?

POR QUE DEVEMOS FALAR EM UMA COMPLEMENTARIDADE DE SENTIDO E REFERÊNCIA?

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ABSTRACT

Until around 1800, Western philosophy believed that there were two types of conception in the world: the mental and the physical. Hence the extensive discussions about the analytical and synthetic knowledge that dominated the philosophy of Kant, the greatest Enlightenment philosopher. However, from the Peircean studies, the discussion about the conceptions has expanded, giving rise to the complementarity, which currently addresses the conceptions of extension and intension_of logic and philosophy. In the educational context it is often claimed that mathematics is a language, since it provides both a means of communication and a substantiation of our thoughts. As a result, mathematical fluidity is now considered the most important. From this perspective, the pedagogical principles underlying mathematics teaching become similar to those used in language teaching. But mathematics is not mere language. Language is a wonderful instrument of the human spirit, yet it serves logic, poetics, and rhetoric far better than mathematics. Thus, this article aims to show that the approach of elementary mathematics education must consist in teaching to read a term beyond its correspondence between letters and sounds, and also to permit the understanding how a skill set can be worked completely in abstract

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in relation to content. The semiotic methodology is utilized as input to analyze what is really the mathematics.

Keywords: Semiotics. Complementarity. Language. Mathematics.

RESUMO

Até por volta de 1800, a filosofia ocidental acreditava que havia dois tipos de concepção no mundo: as mentais e as físicas. Daí, as extensas discussões sobre o conhecimento analítico e sintético que dominou a filosofia de Kant, o maior filósofo do Iluminismo. Porém, a partir dos estudos peirceanos, a discussão sobre as concepções ampliou, dando origem à complementaridade, que, atualmente, aborda as concepções de extensão e de intensão da lógica e da filosofia. No contexto educacional frequentemente se afirma que a matemática é uma linguagem, uma vez que ela fornece tanto um meio de comunicação quanto uma substanciação dos nossos pensamentos. Como consequência, a fluidez matemática passa a ser considerada a mais importante. Nessa perspectiva, os princípios pedagógicos subjacentes ao ensino da matemática se tornam semelhantes aos utilizados no ensino de línguas. Mas, a matemática não é mera linguagem. A linguagem é um instrumento maravilhoso do espírito humano, contudo serve muito melhor à lógica, à poesia e à retórica do que à matemática. Dessa forma, este artigo objetiva mostrar que a abordagem da educação matemática elementar deve consistir em ensinar a ler um termo além da sua correspondência entre letras e sons, e também em permitir a compreensão de como um conjunto de habilidades pode ser trabalhado completamente de forma abstrata em relação ao conteúdo, abrangendo a complementaridade de intensão e extensão. A metodologia semiótica é utilizada como aporte para analisar sobre o que é realmente a matemática.

Palavras-chave: Semiótica. Complementaridade. Linguagem. Matemática.

1 INTRODUCTION

The illustrious mathematician Reuben Hersh (1927-2020), already questioned, in 1997, *What is Mathematics really?* in his book with the same name. He proposed to consider mathematical objects as social entities and to recognize that mathematics is an essentially social reality with intention in to avoid the alternative of idealism versus empiricism. Social entities, he said: “have mental and physical aspects, but none is a mental or a physical entity” (1997, p. 14). For him, questions about the nature of mathematical objects could only be answered from a social perspective. From this understanding we can conclude that the concept of a theory has evolved in parallel with the changes in our view of society.

An example of a changing values of society can be characterized by the French Revolution. Louis XVI (1754 -1793), the last King of France before the fall of the monarchy, could not accept how Robespierre (1758-1794) and the Jacobins could put him on trial, because traditionally the king himself was the state and the law. How can one bring the king’s case to court, if he himself embodies the law?

The modern states and modern societies emerged in the aftermath of the French Revolution and were characterized by individualistic ideals (freedom, equality), on the one

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hand, and a formal structure of laws and the legal systems, on the other hand. The Code Civil des Français as well as the Code penal transformed traditional society which until then was organized around formal social stratification such as caste or class into modern society.

The formal structure of laws was a product of opinions adopted by Romantics in France and Germany that transformed knowledge, and especially mathematics, as much as opinions about our place in society and in the universe. For lack of space, we quote two witnesses, Novalis (1772-1801)[6] and Georg Hamann (1730-1788), respectively.

Novalis:

> The designation by tones and strokes is an admirable abstraction. Four letters signify *God* to me; a few strokes a million things. How easy is the handling of the Universe, how vividly the concentricity of the spiritual world! Language theory is the dynamics of the spiritual kingdom. A command moves armies; the word *freedom* nations (1960, p.412, our translation).

Hamann:

> A law is never as disturbing and insulting as a verdict based on convenient approval. The first does not touch my self-esteem at all, and extends to my action alone, therefore equates all those who are in the same situation. An arbitrary decision without a law is always a bondage for us (1988, p. 59, our translation).

The language for Hamann is exactly the opposite of what was affirmed by the linguistic theories of the Enlightenment. Both Novalis and Hamann emphasized the creativity of language and symbolism, since everything seems to be dissolved in language or semiotics in general.

The view of mathematics as a language has been especially emphasized among scholars of logic, the humanities as well as in educational contexts. According to Edward Effros (1935-2019), even if the premise is that mathematics is a language, since it provides both a means of communication and a substantiation of our thoughts, and even though this aspect of mathematics explains its role Fundamental to modern science, the argument that we should focus on teaching problem solving methods represents a basic misunderstanding about the purpose of mathematical education. After all, we don’t include algebra in the high school curriculum to allow students to solve word problems (1998).

Norbert Wiener (1894-1964) elucidates these transformations from a functional view to theoretical structuralism by characterizing the new intellectual individualism by stating that:

[6] Georg Philipp Friedrich von Hardenberg (1772-1801), scientifically known by the pseudonym *Novalis*, was one of the most important representatives of German romanticism in the late 18th century.

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> […] he who concentrates on his own mental states will concentrate, when he becomes a mathematician, on the proof of mathematical theorems, rather than on the theorems themselves, and will be compelled to object to inadequate proofs of adequate theorems. […] To us, nowadays, the chief theme of the mathematicians of the Romantic period may sound most unromantic and repelling. The new mathematics devoted itself to rigor, […] What the new generation in mathematics had discovered was the *mathematician*; just as what the Romantics had discovered in poetry was the poet and what they had discovered in music was the musician (WIENER, 1951, p. 92-96).

This article will argue against the paradigm that mathematics is a language and in favor of it being an activity that changes according to the conceptions established by society. The methodological approach is based on peircean semiotics, with the contribution of a theoretical analysis that seeks to demystify the reality of mathematical objects.

2 WORDS AND THINGS

Before the Scientific Revolution of the 17th century, classical knowledge was completely determined by its object. To think meant to think about one's own being. The nature of scientific thought was in the very understanding of what existed. "The Aristotelian logic, in its general principles, is a true expression of the Aristotelian metaphysics" (CASSIRER, 1953, p. 4), and the methods of investigation have always had to be congruent with the objects investigated.

Then, at some point in history, it happened that words and things separated and the common interpretation of our sensory impressions seemed to become totally unreliable. "Since the late 16th century, more and more authors opt for the certainty of the method and the mathematical method therefore gains in importance, because it is the safest" (SCHÜLING, 1969, p. 76, our translation). It is a merit that Michel Foucault (1926-1984) brought this concept to the center of our attention. At the beginning of the 17th century, Foucault stated:

> […] writing has ceased to be the prose of the world, resemblances and signs have dissolved their former alliance; similitudes have become deceptive. […] Thought ceases to move in the element of resemblance. Similitude is no longer the form of knowledge but rather the occasion of error, […] 'It is a frequent habit', says Descartes, in the first lines of his *Regulae*, 'when we discover several resemblances between things, to attribute to both equally, even on points in which they are really different, that which we have recognized to be true of only one of them'. The age of resemblance is drawing to a close. […] And just as interpretation in the sixteenth century […] was essentially a knowledge based upon similitude, so the ordering of things by means of signs constitutes all empirical forms of knowledge as knowledge based upon identity and difference (1973, p. 47-51 e p. 56-57).

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Several authors pointed out that the main impact of the Scientific Revolution of the 16th/17th century came from a change in habits of thought and, in particular, from a campaign for individual cognitive certainty. It was the central problem of René Descartes (1596-1650) and the general objective of his *Discourse on the Method*. "I was especially delighted with the mathematics because of the certitude and evidence of their reasonings. But in the beginning, I did not realize its genuine usefulness, thinking that they had but contributed to the advancement of the mechanical arts" (DESCARTES, 2001, Part I).

There was, however, a second problem of knowledge caused by changes in social relations. A person is a manifestation of social existence within a space of possibilities and situated within a world and a cultural tradition. Hence the controversy between the analytical perspective of say Gottfried Leibniz (1646-1716) and the intuitive and geometrical view of Descartes.

According to Ian Hacking (1936) "Leibniz was sure that mathematical truth is constituted by proof, while Descartes thought that truth conditions have nothing to do with demonstrations" (1984, p. 211). This does not mean that Descartes was less concerned with truth or certainty. On the contrary: "The one activity in the world, which really does concern Descartes, is thought and the pursuit of truth. Had he composed the Lord's Prayer, it would no doubt have contained the invocation 'and lead us not into error'" (GELLNER, 1992, p. 7).

An arguably important personification of Cartesian individualism, as well as Leibniz's social concerns, can be found in the Protestant movement and its principle of *Sola scriptura*. Martin Luther (1483-1546), having been summoned by Emperor Charles V to renounce his religious teachings, demanded that his errors be proved in the Bible. The text of the Bible was the only authority to which he would bow. The young emperor was genuinely shocked. If it were granted to defend oneself in the light of the scriptures to those who contradict the advice and common understanding of the State and the Church, then "[...] will have nothing in Christianity that is certain or decided" (RYRIE, 2017, p. 28).

For Leibniz in the centuries 16th/17th mathematics or logic was an art of producing proofs which should convince people to accept something and to educate them. Leibniz did not follow the *Sola scriptura* principle but relied rather on logical proofs of God's existence.

It is this transformation that Foucault has called the transition from a time of *Interpretation* to the age of *Representation*. And, thought is a sign, is a doctrine on which Leibniz and the "[...] thinkers of the years about 1700" all agree (PEIRCE, CP 5.470). The idea

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of calculating with the famous unknown *X* of common algebra exemplifies the new open role signs had acquired. Since then semiotics has become a centerpiece of epistemology and mathematics is essentially an activity operating on symbols and diagrams, semiotics became a fundamental concern.

From semiotics, the way of perceiving ideas changed completely. Now, we don't need to defend one idea at the expense of another that is contradictory. Considering the contradictory examples between Leibniz and Descartes referenced above, it is possible to see that both had their reasons and their limitations. Currently, it is necessary to know the context of each situation to choose the best tool, the best path, the best truth. The semiotic perspective showed us that both Leibniz's analytical thinking (reference) and Descartes' intuitionism (sense) are important for the evolution of mathematics.

To reinforce the understanding of the changes that occur over time and how they implied changes in the sciences, including mathematics education, we will present, next, transformations that occurred in the interpretation of geometry and algebra, through the development of some concepts, ideas, theories, perceptions and foundations that supported the sciences, as well as portray some implications in the mathematical sciences.

3 TRANSFORMATION IN THE INTERPRETATION OF GEOMETRY

The historical development of mathematics is conceived as a sequence of symbolic or linguistic innovations. In Euclid's *Elements*, for example, one praises the scientific mind that appears in the organized arrangement of problems and theorem, but never realizes that Euclid's geometry is a theory of figures in a metaphysical no-man's-land and does not include a theory of space. Therefore, certain problems, like the problem of the duplication of the cube, for example, cannot be solved within the context of Euclidean geometry. The problem of the duplication of the cube, for example, can be solved easily however, using the principle of continuity. But the Greeks never arrived at the insight that in mathematics it might be conceivable to have a solution, that is totally divorced from constructability.

The *Copernican Revolution of Epistemology* by Immanuel Kant (1724-1804) brought the idea of space, conceiving it as a kind of subjective means. "Space is not an empirical concept. […] Space is a necessary representation, *a priori*, which serves as the foundation of all external intuitions" (KANT, 1787, B39). Kant's use of the idea of space is often criticized

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in circles of analytical philosophy, however, it is criticized for the wrong reasons, namely for an insufficiency of its logic. Here is what Milton Friedman (1912-2006) writes:

> Kant's conception of mathematical proof is of course anathema to us. Spatial figures, however produced, are not essential constituents of proofs, but at best aids (and very possibly misleading ones) […] The proof itself is a purely formal or conceptual object, ideally a string of expressions in a given formal language (1992, p. 58).

For Friedman, mathematics is a matter of logic. According to analytical philosophy, formal mathematics and logic do not talk about objects. "They say nothing about objects, of which we want to speak, but deal only with the way we talk about objects" (HAHN, 1988, p. 150). Thus, the proofs of classical geometry should perhaps be considered more appropriately in analogy to mental experiments in the natural sciences.

According to Ian Mueller (1938-2010):

> Part of the difficulty is due to a failure to distinguish two ways of interpreting general statements like 'All isosceles triangles have their bases angles equal'. Under one interpretation the statement refers to a definite totality [...] and it says something about each one of them. Under the other interpretation no such definite totality is presupposed, and the sentence has much more conditional character – 'If a triangle is isosceles, its two base angles are equal' (1969, p. 291).

Thomas Kuhn (1922-1996) argues, in *A Function for Experiments of Thought* (1977), that the productivity of the thought experiment is due to its function of readjusting the relationship between a conceptual apparatus or a theory and the reality to which it is applied. Therefore, mental experiments can teach us something new about the world, although we have no new data, it helps us to reconceptualize the world in a better way.

Considering what was exposed in B744 of Kant's *Critique of Pure Reason* (1787), about the construction of his proof of the sum of the angles of a right triangle theorem, the following mental experiment seems, in fact, to reveal some limitations of the proof de Kant: Suppose we go through the perimeter of a triangle. How many degrees do we turn after we return to our starting position? Simple answer: 360 degrees, because our entrance direction coincides with the final one. This answer, however, while intuitively convincing and obvious, is based on the assumption that it is, on the one hand, the same as rotating the site by a total angle of 360 degrees or, on the other hand, doing this by passing through a closed line at the perimeter of an arbitrarily large triangle.

One case, however, is based on local characteristics of space, the other is not! For arbitrary triangles our conclusion is only valid in the Euclidean plane, but is invalid on the

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surface of the sphere, for example, as everybody may perceive for himself quite easily. And Kant knew that we humans live on a sphere. Spherical geometry is a generalization of Euclidean geometry. Non-Euclidean geometries became accepted only after Eugênio Beltrami (1835-1900) had proved their consistency in 1868. The ordinary sphere was received as a model of a space with positive curvature.

Euclid deals with the angle-sum theorem of the triangle in proposition 32 of Book I: “In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles”.

The proof of this proposition makes use of propositions 13, 29, and 31, which in turn rely on propositions 11, 13, 15, 23 and 27, and so on, back to the postulates. This structure is not based on a logical-deductive connection, but it arises from the activity of solving plane geometric problems. And as such it shows closer affinity to Luitzen Egbertus Jan Brouwer (1881-1966) intuitionism than of is David Hilbert (1862-1943) axiomatic approach. In fact, Andréi Kolmogorov (1903-1987), one of the greatest mathematicians of the 20th century, interpreted Brouwer’s intuitionistic logic in terms of problems and solutions. To assert a formula is to claim to know a solution to the problem represented by that formula. For instance, *P implies Q* is the problem of reducing *Q* to *P*; to solve it, requires a method to solve problem *Q*, given a solution to problem *P*. Kolmogorov writes:

> In addition to the theoretical logic which systematizes the proof schemes of the theoretical truths, one can also systematize the solutions of problems, e.g. of geometric construction problems. In analogy to the principle of syllogism, the following principle holds here: if we can reduce the solution of *b* to the solution of *a* and the solution of *c* to the solution of *b*, then we can also reduce the solution of *c* to the solution of *a*. […] The following remarkable fact applies: According to its form, this task calculus coincides with Brouwer’s intuitionist logic, as formalized by Mr. Heyting (1932, p. 58, our translation).

In this way, mathematical objects are established by the identity relation chosen. Since the 16th century at least *congruence* of plane figures was chosen as the most distinctive geometrical equivalence relation. What about Euclid and the Greeks? In the very short argument of §35 (theorem 25) of book I of Euclid’s *Elements* the word *equal* occurs more than 10 times, with three different meanings: *congruence* of flat figures, *equality of area* and *numeric identity*. The theorem reads: “The parallelograms which are on the same base and in between the same parallels equal one another”.

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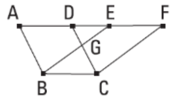

Figure 1 – Diagram of Theorem 25
Font: Euclides (2009, p. 124)

The question arises what Euclid means in using the word *equal*? David Fowler (1937-2004) says that the: “[...] the idea behind Euclid’s use of equality within geometry is one of size not one of shape and his concern is to see if two plane figures are equal in size” (1987, p. 13).

This contradicts the common view of our textbooks, which since Leibniz already defines geometrical equality in terms of congruence. Fowler’s evaluation and judgment are supported by a number of passages from the Platonic dialogues. In the *Republic* we read for example:

> Now no one with a little experience of geometry will dispute that the science is entirely the opposite of what is said about it in the accounts of its practitioners […] They speak like practical men and all their accounts refer to doing things. They talk of squaring, applying, adding and the like, whereas the entire subject is pursued for the sake of knowledge. (PLATO, 1997, Rep. VII, 527 b).

In this context, Euclid’s triangles or rectangles etc. are diagrams, that is, they are signs, in which the meaning or reference or extension is the thing itself. However, what the triangle represents is the world to which it belongs, and that world has undergone changes. Since the 17th century, this world has been the world of science or theory as part of a science. Consequently, the sum of the angles of the triangle theorem would have to be analytical, which in fact it is, according to formal axiomatic geometry like that of Hilbert or Giuseppe Peano (1858-1932). According to the classical understanding of concepts or ideas, the order of extensions inverts the order of ideas. This led Leibniz, his contemporaries and his successors to define geometric equality in terms of congruence and not in terms of material identity.

4 TRANSFORMATION IN THE INTERPRETATION OF THE ALGEBRA

In the history of algebra, something similar can be noted Leonhard Euler (1707-1783) in *Complete Guide to Algebra* (1770) begins by introducing the notion of quantity and then that “arithmetic or the art of calculation deals with the numbers, it only affects activities in ordinary life, in contrast, algebra or analytics generally includes everything that can occur with numbers and their determination” (EULER, p. 5, chapter 1, §7, our translation).

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All the difficulties that the Greeks had with the fact that the numbers are not actually quantities, are circumvented by identifying the magnitudes with their measurements, instead of considering the numbers as relationships between the measured magnitude and the scale used. As in geometry, this philosophical setback allowed for some progress on the active side of mathematics. This reduction in object relations was, in a sense, the result of Descartes' view of analytical geometry. In this way, it always operates on a symbolic level and all progress has been attributed to the invention of new methods and new symbols.

The story of so-called imaginary numbers is an example of this. These imaginary entities have brought great progress to the treatment of algebraic equations. But, while the imaginary unit gained acceptance of arithmetic only as a symbol for calculating, it also produced some strange confusions (NAHIN, 1998). Only after Carl Friedrich Gauss (1777-1855) presented a geometric interpretation to the imaginary unity of the model called the numerical-Gaussian plane, did it become a legitimate mathematical object, which later assumed an important role in mathematics and metamathematics.

Only then algebra was no longer conceived of as an analytical language but as a theory of formal structures. The notation $a + b\ (2)^{1/2}$ or if we abbreviate $(2)^{1/2}$ by t: $a.1 + b.t$ for the enlarged set of algebraic numbers suggests the idea of the concept of vector space (EULER, 1770), because 1 and t are linearly independent vectors over the rationals, in exactly the same way, as 1 and i are linearly independent vectors over the reals. However, nobody saw and explored that analogy before the 19th century. Only after modern mathematics had discovered the complementary notions of *set* and *structure* pure mathematics developed.

The scientist John David Barrow (1952-) still characterizes the algebraic spirit today in his book *Perche il mondo e matematico?* (1992) as follows:

> The mathematical language acts like a computer language, because it is primarily a language with a built-in logic. We know that we need not be so particular with ordinary language. If we do not keep strictly to the rules of grammar and syntax, we are nevertheless understood. But if we do not adhere to the rules of mathematical language, everything becomes meaningless. Frequently, students are instructed that they must think about things in order to understand them and to move forward. But in some sense, the greatest progress of human thought has incurred as a result that we have learned to do things without thinking (BARROW, 1992, p. 3, our translation).

Jean-Victor Poncelet (1788-1867) identified the secret of algebraic generality elsewhere, claiming that it is as due to relational thinking and in particular to the principle of continuity. Friedrich Ludwig Gottlob Frege (1848-1925) expressed a similar idea by saying that

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the ascent from arithmetic to algebra is due to functional thinking and based on the introduction of the function concept (FREGE, 1969).

If we write down the following sequence of arithmetic expressions: $2.1^3 + 1$, $2.2^3 + 2$, $2.3^3 + 3$ and so on, we could come up, concentrating on the common form, with the expression: $2.x^3 + x$. That is, we might encounter the idea of a *function*.

In this way, Frege reintroduced relational thinking into arithmetic, as the concept of function had already been known to mathematicians through Descartes' theory of analytical geometry that constructed functions or curves, rather than geometric figures. Only through the concept of function did the notion of algebraic variable as well as the structural view of mathematical theory gain entry into mathematics. For this reason, Descartes can be said to be the first to really have a deep understanding of relational thinking. Probably Leibniz had already recognized this, however, he had criticized Descartes for not radically following the consequences from a formal axiomatic perspective.

The concept of a mathematical function, on which the notion of natural law is based, "[…] applied to physical phenomena, appeared for the first time in the literature of mankind in a prescription for gunners in 1546" (ZILSEL, 2003, p. 110), eighteen years before the birth of Galileo Galilei (1564-1642) and exactly half a century before the birth of Descartes.

The difference is that between an intensional (meaning) resp. extensional (reference) conception of a function (FONSECA, 2010). Either the function is identified with an *algorithm* or with *some kind of free variable as part of a law of nature* (intensional conception) or like in the famous general triangle or like in expressions like *An apple is a fruit* (extensional conception).

In a proposition like *an apple is a fruit* it would be unnatural to interpret *an apple* as a placeholder, like Frege, because this presupposes that we have given individual names to all the apples in this world (QUINE, 1974). There are ideas of an apple or a triangle in general, or of a function but they turn out to be representations of particular ideas, put to a certain use.

Pierre Boutroux (1880-1922), Jules Henri Poincaré (1854-1912) or Charles Sanders Peirce (1839-1914), have adopted the intensional (meaning) perspective on functions, taking a function as a concept in its own right, while Georg Ferdinand Ludwig Philipp Cantor (1845-1918), Bertrand Arthur William Russell (1872-1970) or Frege adopted the extensional (reference) perspective of functions, reduced functions to sets (OTTE, 1990).

But as was said the Cartesian analogy between arithmetic and geometry accomplished both the identification of magnitudes with numbers as well as the geometrical interpretation of

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algebraic equations. Descartes had in 1619 already tried to design a program and a method by which the problems of continuous and discrete magnitude could be treated analogically (ADAM; MILHAUD, 1936).

The classical example that clearly demonstrates the complementarity of arithmetic and geometry or of the discrete and the continuous comes from Zeno's paradox of Achilles and the tortoise. And this paradox could be resolved by passing from a set-theoretical view of the continuum to an intensional one, using the function concept. Let us see: Achilles runs ten times as fast as the tortoise, though the tortoise has a one stadium start. For *each* of the stages, $x(x > 0)$, covered by Achilles, the tortoise has crawled the distance $f(x) = 1/10x + 1$ *stadium.*

This function as a model of the movement, or rather the relative movement of the tortoise to the *standing* position of Achilles, now enables us to reproduce the paradox on a new level because of its double character: the continuous aspect of the movement does not contradict the discrete perspective. It remains correct that the tortoise is at $x_{(n+1)}$ as soon as Achilles has reached x_n. But the representation using the function concept enables us to liberate Achilles' movement from the one-sided fixation on the discrete x_i $(i = 0, 1, ...)$, seeing the movement as a whole.

That is, the relative movement of Achilles and the turtle is a linear function, because both movements are uniform: $f(x) = ax + b$ (that is, when Achilles reaches x, the turtle is at $f(x)$).

The question *At what point does Achilles really catch up with the tortoise?*, becomes now: *What is the fix point of f(x)?* The fixed point can be calculated simply as a function of the constants a and b: $x = f(x) = ax + b$. We seemingly have solved the problem by taking a relational point of view, that means by adopting a *world view* which provides objects and relations between objects with an equal ontological status.

This essentially constitutes what has been called the transition from thinking about objects to complementary relational thinking. This transition took place only at the end of the 18th century. In what sense is this a solution? The paradox of the movement leads to a complementarity in the *function* concept! It shows the necessity of having the concept of the functional relation as a model or as a single mathematical object. And secondly, to have available the effectiveness of symbolic calculations, that allow us to write down the meeting point (OTTE, 1990).

In this sense, mathematician Salomon Bochner (1899-1982) states:

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> Functions are a distinguishing attribute of modern mathematics, perhaps the most profoundly distinguishing of all. […] In its innermost structure Greek mathematics was a mathematics entirely without functions and without any orientation towards functions. […] By outward appearance Greek mathematics was geometrical rather than analytical and by inward structure it was representational rather than operational (1966, p. 217).

5 MATHEMATIZED SCIENCE

Jacob Klein (1899-1978) introduces his fundamental study on the transformation of the Hellenistic conception of mathematics and science with the following words:

> The creation of a formal mathematical language was of decisive significance for the constitution of modern mathematical physics. If the mathematical presentation is regarded as a mere device, preferred on only because the insights of natural science can be expressed by symbols in the simplest and most exact manner possible, the meaning of the symbolism, as well as of the special methods of the physical disciplines in general will be misunderstood (1992, p. 1).

Let us observe in passing that the mathematization of natural science introduces representations, which even may contradict the physical facts. For example, Thomas Kuhn indicates that the term *Mass* has different meanings in classical Newtonian Mechanics and in Einstein's Special Theory of Relativity: the Newtonian mass is stable, independent of velocity, whereas the Einsteinian one depends on the velocity.

Considering the expression: $m = m_0 / (1- v^2/c^2)^{1/2}$, where m and m_o are the mass and the initial mass, v is the body velocity and c is the light velocity, when we assume that the light velocity c passes to infinity, we get $m = m_0$. However, this *passing to infinity* is explicitly forbidden by the physical facts, as they became known through the *Morley-Michelson*-Experiment. And it were these facts that stimulated Albert Einstein (1879-1955) of is relativity theory in the first place.

Another interesting example where the formal mathematical representation contradicts physical reality is the following: if you want to change the conditions in an electrical network, you need different switches, which change the flow of electricity. One result of the formal automaton theory is that, regardless of the size and complexity of the network, one always manages with two switches of a certain type to reverse all directions of current flow in the lines. In the 1950s, on the other hand, mathematicians such as Andrei Andreyevich Markov (1856-1922), who is a pioneer of constructive mathematics, have shown that contrary to mathematical representation, the number of inversion switches is not independent of the number of input and

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output channels of the network. Lee Cecil Fletcher Sallows (1944-) has analyzed this contradiction between formal proof and objective reality and found that in the real machine certain feedbacks take place, due to the inertia of material systems. A current flow cannot be instantaneously be interrupted or vice versa. This ultimately leads to different effects than appear in the static diagram of mathematics. And Sallows concludes, that notion that everything can always be talked about in a different language is thus not without its pitfalls (1990).

Therefore, we can see again that the mathematics of natural science depends on a relative independence of meaning and reference.

Mathematics deals with objects, but these objects belong to some model world, to some limited universe of discourse, because mathematic is no empirical science, like physics or biology. Therefore, we have to create model worlds. And this fact provides the intensions and extensions of mathematical terms with equal importance, such that mathematics or logic are not merely formal languages.

As an example, one might remind oneself of the intense debates between Frege and Hilbert or Russell and Peano. One topic in these discussions concerned the axiomatic presentation of number. Russell's criticism was that the axiomatic characterization of number, led to a situation where: "[…] number-symbol becomes infinitely ambiguous. […] we want our numbers not merely verify mathematical formulae, but to apply in the right way to common objects" (RUSSELL, 1998, p.9). Russell seemed, however, not to have perceived clearly that formal axiomatics was a very general method of mathematics and that it was therefore in need of a some applications to establish its status as real knowledge. Now pure mathematics wanting to secure its autonomy as a profession and to free itself from all applications, choose set-theoretical foundation as a substitute. But strictly formalistic mathematics, as it was developed by Hilbert's school, did at first not pay sufficient attention to the burden of set-theoretic tools, which were strictly connected with axiomatics.

There is another problem with the algorithmic or linguistic view of mathematics: you cannot perform impossibility proofs, such as doubling of the cube, trisecting of the angle, etc., which are, however, a kind of birth certificate of modern mathematics and culminate in the work of Friedrich Gödel (1906-1978). The proof of the impossibility of doubling the cube with Euclidean means, for example, became possible as soon as people modelled the geometrical constructions in arithmetical terms, creating the notion of *constructible number* and finally showed than the third root of *2* was not a constructible number.

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6 CONCLUSION

When we encounter something new and completely unknown, the only thing we can do is to represent it by some arbitrary symbol or name. We can even represent the completely unknown, using some index as in the case of the famous *x* of symbolic algebra, or in the context of the hypothetical deductive reasoning of axiomatic theory.

On many occasions it is not sufficient to have an idea. One has to act and to apply it. An algebraic equation should be solvable, a theory should be applicable, a machine should work, and a scientific concept is essentially a function. A lamp that gives no light, a knife that does not cut, a corkscrew that does not pull out the cork, they all are useless, are nothing. Mathematical or scientific representations gain their significance only in the application. Formal axiomatic structures or theories are instruments in the same way as maps or diagrams or symbols, etc.

The truth and the foundations of an axiomatic theory lie therefore in the future in the intended applications. Any formal axiomatic theory possible has quite a number of different intended applications. What the axioms describe are concepts or classes of objects, rather than particular objects themselves. Peano's axioms do not answer the question: *What are numbers, what is the number 1 or 2?* Numbers could be anything, even games (Conway-Numbers, Hackenbusch-Games, Chessboard-Computer, etc.).

Thus, as has already been said, modern mathematics or theoretical science speak in terms of complementary notions of *set* and *structure*. The objectivity and operational development of the concepts are interconnected. This interest in the operative fertility of scientific and mathematical concepts grew enormously during the *Scientific Revolution* that began in the 16th century and continued until the 18th century.

Descartes' concern concerns the issue of internal representation – an idea – that provides us with the information. The similarity between representation and represented will not be maintained, given the nature of the causal interaction between the observer and the world. Descartes believed more in problem solving than in building formal theories and evidence. Leibniz believed in formal theory and logical identity. Leibniz invented formal proof as we know it today (HACKING, 1984).

Leibniz, however, obtained the essential ideas of the arithmeticization of Descartes' geometry, which was, at the same time, also a geometrization of arithmetic. He criticized Descartes for not going far enough in the search for the first axiomatic foundations of

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knowledge. Descartes was a geometer and was not very fond of arithmetic or algebra, and Leibniz was a formalist and algebraic. Descartes' intuitions were personal and could not be taught and communicated. However, they are more fertile than any formal proof. In contrast, Ian Hacking writes:

> We have usually read him as an ego, trapped in the world of ideas, trying to find out what corresponds to his ideas, and pondering questions of the form, 'How can I ever know?' Underneath his work lies a much deeper worry. […] one is led, I think, to a new kind of worry. I cannot doubt an eternal truth when I am contemplating it clearly and distinctly. But when I cease to contemplate, it is a question whether there is truth or falsehood in what I remember having perceived. Bréhier suggested that demonstrated propositions may go false […] They exist in the mind only as perceptions. Do they have any status at all when not perceived (1984, p. 220-221).

Hacking makes a mistake, because scientific or mathematical knowledge is *social* knowledge. Descartes, as a mathematician, was satisfied when he was able to solve his problems and was not interested in building universal theories and demonstrating eternal truths. We must remember that the same divergences still prevail today.

William Timothy Gowers (1963-), renowned mathematician and medalist Fields (1998), addresses a similar situation in today's mathematical culture, identifying two different cultures in fact:

> The *two cultures* I wish to discuss will be familiar to all professional mathematicians. Loosely speaking, I mean the distinction between mathematicians who regard their central aim as being to solve problems, and those who are more concerned with building and understanding theories (2000, p. 65).

Gowers sees himself as a problem solver, identifying Michael Francis Atiyah (1929-2019), another Fields medalist of (1966) as a theoretician and as his counterpart.

The duty of language in all areas of human self-reflection is to repeat the same things in ever new ways. Only thereby can Man try to become aware of himself and of his destiny. However, mathematics is not a language and here applies what Sallows had observed in 1900, namely that the notion that everything can be talked about in a different language is not without its pitfalls.

The elementary mathematical education approach consists, not only, of reading a term in addition to its correspondence between letters and sounds, but also to understanding how a set of skills can be worked in a completely abstract way to the content, encompassing the complementarity of meaning and reference. The epistemological historical context in

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Foucault's words and things, the transformations in the interpretation of geometry and algebra and the mathematics of natural science, exemplify a relative independence of meaning and reference. Culminating in the importance of discussing and studying more closely the complementarity of meaning and reference in mathematical education.

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