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CONCEPT IMAGE AND CONCEPT DEFINITION SCHEMES AS TOOLS FOR ANALYZING STUDENTS' COMPREHENSION OF LIMIT OF A FUNCTION

ESQUEMAS DE IMAGEM E DEFINIÇÃO CONCEITUAL COMO FERRAMENTAS PARA ANÁLISE DA COMPREENSÃO DE ESTUDANTES SOBRE LIMITE DE UMA FUNÇÃO

ESQUEMAS DE IMAGEN Y DEFINICIÓN CONCEPTUAL COMO HERRAMIENTAS PARA ANALIZAR LA COMPRENSIÓN DE LOS ESTUDIOS SOBRE EL LÍMITE DE UNA FUNCIÓN

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ABSTRACT

This paper aims to analyze one's understanding of limit of a function based on Concept Image and Concept Definition schemes, which can be used as powerful tools in research that have mathematical comprehension as part of their object of study. For that, we developed and applied a set of tasks involving limits of functions to students who had already been in a Calculus course. Vinner's study about concept image and concept definition was the theoretical framework of this paper. The results obtained let us observe the plurality of interpretations about limits and other adjacent concepts, which allowed us to trace, in this paper, a reflection upon the cognitive conflicts concerning the (lack of) understanding of these concepts.

Keywords: Concept Image. Concept Definition. Schemes. Limit of a Function.

RESUMO

Este artigo tem como objetivo analisar a compreensão de estudantes sobre limite de uma função com base em esquemas de Imagem Conceitual (IC) e Definição Conceitual (DC), que podem ser utilizados como ferramentas poderosas em pesquisas que tenham a compreensão matemática como parte de seu objeto de estudo. Para isso, desenvolveu-se e aplicou-se um conjunto de tarefas envolvendo limites de funções a alunos que já haviam cursado um curso de Cálculo. O estudo de Vinner sobre IC e DC se configurou como arcabouço teórico deste artigo. Os resultados obtidos permitiram observar a pluralidade de interpretações sobre limites e outros conceitos adjacentes, o que possibilitou que fosse traçado, neste artigo, uma reflexão sobre os conflitos cognitivos relativos à (falta de) compreensão desses conceitos.

Palavras-chave: Imagem Conceitual. Definição Conceitual. Esquemas. Limite de uma Função.

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RESUMEN

Este artículo tiene como objetivo analizar la comprensión de los estudiantes sobre el límite de una función con base en esquemas de Imagem Conceitual (IC) y Definição Conceitual (DC), que pueden ser utilizados como herramientas poderosas en pesquisas que tienen una comprensión matemática como parte de su objeto de estudio. Para esto, desenvolveu-se-se e aplicou-se um conjunto de tarefas involucrando limites de funções a alunos que já haviam cursado um curso de Cálculo. El estudio de Vinner sobre IC y DC se configura como arcano teórico de este artículo. Los resultados obtenidos permitirán observar una pluralidad de interpretaciones sobre límites y otros conceptos adyacentes, o que possibilitou que fosse traçado, neste artigo, una reflexión sobre los conflictos cognitivos relativos a (falta de) comprensão desses conceitos.

Palabras clave: Imagen Conceptual. Definición Conceptual. Esquemas. Límite de una función.

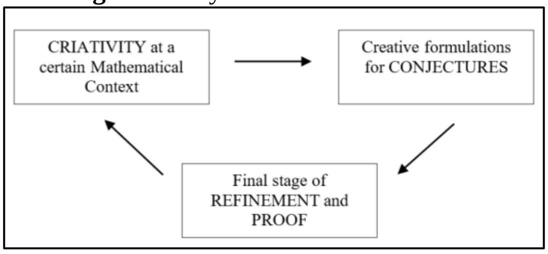
1 INTRODUCTION

The Psychology of Learning plays an essential role in Mathematics Education, since it is through the comprehension of cognitive processes that one can reflect about practices that may enable an individual to understand a certain mathematical knowledge. In that regard, a lot has been discussed about the learning of mathematics under the psychological dimension of its processes and mental mechanisms, bringing with it reflections on cognitive development and mathematical thinking (Messias; Brandemberg, 2023).

According to Tall (2013), the process of cognitive development involves comprehensions that allow one to think about essential ideas of a concept, in which language plays a facilitating role. In this sense, it is believed that when an individual is ahead of a specific mathematical situation, his/her mind mobilizes multiple connections that constitute a whole Knowledge Structure, composed by a plurality of associated perceptions, ideas and actions that are extremely important for the learning of mathematics at any level.

Concerning to the nature of Advanced Mathematical Thinking (AMT), which it is assumed in this paper to be tied to formal mathematics as well as to a cycle of activities, such as the one represented in figure 1.

Figure 1 – Cycle of activities & AMT



Source: Messias; Brandemberg (2023)

Such cycle does not happen in the same manner to all individuals (Tall, 1991). That is because there are plural ways of thinking about mathematics, either as a mental activity or as a formal system. Among the theoretical frameworks of AMT that discuss about how one comprehends a mathematical concept, it is highlighted in this paper Vinner's (1991) perspective on Concept Image (CI) and Concept Definition (CD). Therefore, the main goal of this work is to analyze one's understanding of limit of a function based on CI and CD schemes, which can be used as powerful tools in research that have mathematical comprehension as part of their object study.

2 CONCEPT IMAGE AND CONCEPT DEFINITION

Vinner (1991) points out that when an individual is faced with a certain mathematical situation, he\she mobilizes several mental processes. His/her mathematical maturation depends on the ability of stablishing connection between such processes and, mostly, on how much his/her memory is stimulated to form non-verbal associations, that is, concept images, based on visual representations, mental figures, processes, and previous learning experiences. It is assumed that to understand a concept means to form a coherent concept image for it.

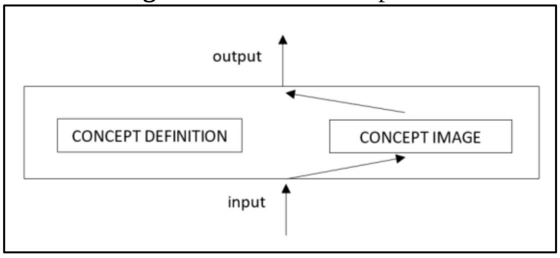
According to this theoretical framework, different parts of one's Concept Image are activated when he\she needs to interpret and/or solve a certain task. Such parts are called Evoked Concept Images (ECI). It is important to mention that an ECI does not necessarily represent all one knows about a concept, since it is possible for him/her to mobilize other elements related to that concept, depending on the mathematical context they are in.

The complexity of an individual's understanding of mathematical objects depends on how mature the connections between the elements that constitute his/her Concept Image are. His/her CI might be translated to a Personal Concept Definition (PCD), that is, to a form of words used to describe an object, which may (or may not) differ from its Formal Concept Definition.

Multiple representations of a concept may form one's Concept Image. However, if, for any reason, the CI is constituted by any misconception, then, it can lead him/her to incoherent responses and, consequently, interfere in one's learning process. The inconsistent parts of an individual's Concept Image remain, unless they are evoked simultaneously (Vinner, 1991), that is when he\she usually realizes the necessity of reformulating any cognitive conflict associated to a mathematical object. Therefore, it is extremely necessary to face contexts of learning that lead them to evoke as many elements of their Concept Image as possible.

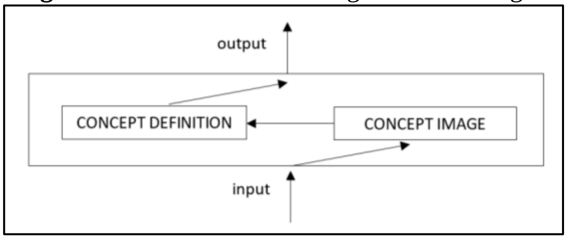
Vinner (1991) presented different kinds of illustrations, from which he shows how CI and CD can be connected to each other when a subject is ahead of a stimulus (mathematical task). The author conjectured about how one's cognitive system might behave when he\she is either solving a problem or building new knowledge, as shown in figures 2 – 5.

Figure 2 – Intuitive Response



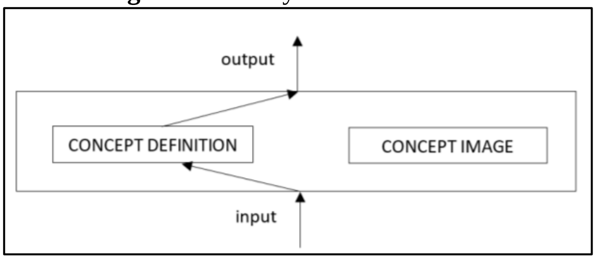
Source: Vinner (1991, p. 73)

Figure 3 – Deduction following intuitive thought.



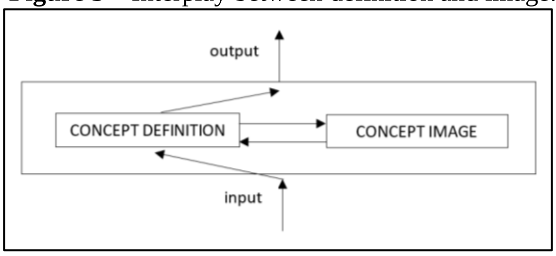
Source: Vinner (1991, p. 72)

Figure 4 – Purely formal deduction.



Source: Vinner (1991, p. 72)

Figure 5 – Interplay between definition and image.



Source: Vinner (1991, p. 71)

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An intuitive response (figure 2) is based on elements that compose an individual's concept image. In this case, there's no conscious relation stablished with formal theory, that is, the concept definition cell is not consulted. Vinner (1991) states that this usually happens because it is not the nature of our cognitive system to consult definitions, either to solve a task or to build new knowledge.

Figure 3 and 5, respectively, illustrate what Vinner (1991) denominates Deduction following intuitive thought (DIT) and Interplay between Definition and Image (IDI). Both are related to the Intuitive Response (IR) due to individual searches, at a certain point, the elements of his/her concept image to respond to what is asked. The principal difference between them is that, in DIT and IDI, it can be noticed a process of deduction, in which the subject searches for information related to (part of) the concept definition cell.

Unlike Intuitive Response, Deduction following Intuitive Thought and the Interplay between Definition and Image, a Purely Formal Deduction (figure 4) does not bring elements from an individual's concept image. It means that, to solve a task, the subject reproduces exclusively de formal concept definition. However, it is assumed in this paper that any response contemplates one's concept image elements, even if it is based explicitly on formal theory. Thus, it is more likely for a subject to behave in the sense of what is shown in figures 2, 3 and 5 whenever he\she is solving a task.

3 METHOD

To show how Concept Image and Concept Definition Schemes are powerful tools for analyzing an individual's mathematical comprehension, it will be presented in this section part of the results of a study developed by the author (Messias, 2018), through which it was discussed how students understood the concepts of limits and continuity of functions of a real variable[1]. The participants needed to solve the tasks shown below:

[1] Undergraduate students of a mathematics course of one university located in the north of Brazil participated of the study. All of them had already taken the Calculus Course.

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1. Consider $f(x)$, and answer what follows.

$$f(x) = \begin{cases} -x+1; & 0 \leq x < 1 \\ 1; & 1 \leq x < 2 \\ 2; & x = 2 \\ x-1; & 2 < x \leq 3 \\ -x+5; & 3 < x < 4 \\ \frac{1}{2}; & x = 4 \end{cases}$$

1.1. "$\lim_{x\to 0} f(x)$ exists". This statement is True or False? Explain.

1.2. Verify if $\lim_{x\to 1} f(x)$ exists. Explain.

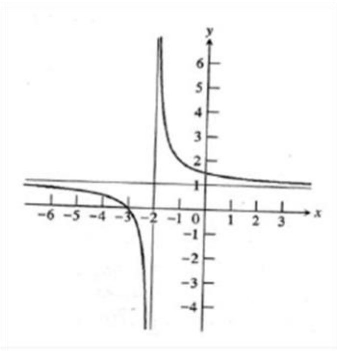
1.3. "$\lim_{x\to 2} f(x)$ doesn't exist". Do you agree with that statement? Explain.

1.4. Determine $\lim_{x\to 4} f(x)$.

1.5. "$\lim_{x\to 3} f(x) = f(3)$". Is this statement True or False? Explain your answer.

2. Explain what you understand of $\lim_{x\to x_0} f(x) = L$.

3. Consider the function $f(x) = \frac{x+3}{x+2}$. Observe its graphic representation, and answer what follows.



3.1. "The straight line $y = 1$ is a horizontal asymptote of $f(x)$". Explain what it means.

3.2. "The straight line $x = -2$ is a vertical asymptote of $f(x)$". Explain what it means.

4.Observe the figures below:

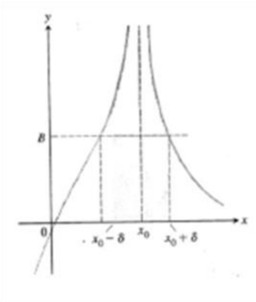


Fig. 4a

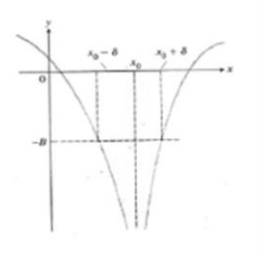


Fig. 4b

4.1. Write a personal definition for $\lim_{x\to x_0} f(x) = L$, based on figure 4a

3.2. Write a personal definition for $\lim_{x\to x_0} f(x) = L$, based on figure 4b.

5. A student verified, while solving a problem, that $\lim_{x\to x_0} f(x) = \frac{0}{0}$. *Explain what such result means. If possible, give examples according to your explanation.*

In task 1, the main goal was to verify if the participants would evaluate and justify the (non) existence of a limit at different points, to check if they would evoke concept images that would put the D_f as a necessary condition for the limit's existence, the same way it was discussed in Przenioslo (2004), Jordan (2005), Karatas et al. (2011), Denbel (2014), and others. Besides that, we aimed to investigate if the function defined in more than one sentence would be a cognitive conflict factor, as noticed in Nascimento (2003), Maharaj (2010), Mutlu & Aydin (2013) and Brandemberg & Messias (2016), and to identify it the participants would evoke the lateral limits to verify the bilateral limit's existence.

In task 2, the participants should explain what they understood of $\lim_{x \to x_0} f(x) = L$. The main goal was to have them write a personal concept definition and to verify if interpretation such as "unreachable" or "impassable" would be evoked, also, if dynamic conceptions related to the nature of that concept would constitute the participants' thought (Tall; Vinner, 1981; Cornu, 1983; Przenioslo, 2004; Sarvestani, 201; Amatangelo, 2013; Messias, 2013; Denbel, 2014).

With task 3, we would like to verify the individuals' understanding of limits involving infinity and horizontal and vertical asymptotes. That is, check if the fact that $\lim_{x \to \infty} f(x) = 1$ and $\lim_{x \to -\infty} f(x) = 1$ would be evoked to explain the asymptote $y = 1$, as well as $\lim_{x \to -2^+} f(x) = \infty$ and $\lim_{x \to -2^-} f(x) = -\infty$ to explain the asymptote $x = -2$. Therefore, we would like to verify students' comprehension of the definition of asymptotes and the way it is related to limits involving infinite. We would also like to evaluate if the students would mobilize the conditions of domain in their answers.

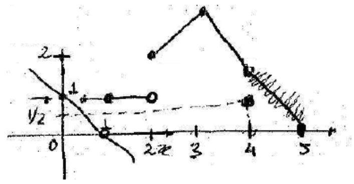
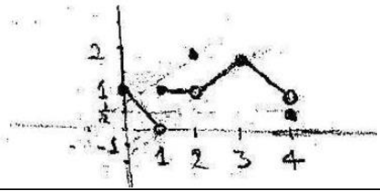
In task 4, it was also asked the students to write a personal definition, considering both, figure a and figure b. We wanted to check if they would evoke the idea of $f(x)$ arbitrarily away from the origin or if they would stablish any relation between the elements that belong to the interval $(x - \delta, x + \delta)$ and their respective values of $f(x)$.

We emphasize that the tasks did not involve the practice of finding limits using algebraic manipulations, since we believe that any difficulties in solving that kind of task are not necessarily related to the nature of the concept, but to the operational practices that are used to find them. Hence, in task 5, instead of finding indeterminate limits, participants were asked to explain what they understood of $\frac{0}{0}$ and, if possible, exemplify situations that agreed with the explanation given. Our main goal was to observe if students would make use of terms such as "removal discontinuity", "points of singularity" or "indetermination" and if they would evoke the idea of nonexistence of the limit attached to the presence of indetermination, as well as in Nascimento (2003), Mutlu and Aydin (2013), Messias (2013), among others.

4 RESULTS & ANALYSIS[2]

Considering task 1, we noticed that two participants drew graphics and used them as a support to answer what was asked. However, only one of them represented the function correctly, as shown in chart 1 (bellow).

Chart 1: Graphic Constructions to Task 1

Graphic of Student S2	**Graphic of Student S4**
	

Source: The author.

S4's responses to task 1 were all correct (and so it was his explanation, which was all done based on the graphic he built). S2 based his responses on the mistaken graphic he drew. As an example, the function $f(x)$ he built was defined for $x < 0$, which made it possible for him to find $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 1$, and consequently, affirmed that $\lim_{x \to 0} f(x) = 1.$ Thus, S2 attached the bilateral limit's existence to the lateral limits (both exist and are the same), i.e., his mistaken graphic was, actually, cognitive conflict factor in task 1.

Three students answered that $\lim_{x \to 0} f(x)$ did not exist, but their explanations were not coherent. S3, for example, replaced $x = 0$ in all sentences that constitute $f(x)$ in its domain, using the fact that all the limits he obtained were different to justify its non-existence, which made it clear for us that evaluating limits of functions written in more than one sentence was a cognitive conflict factor, such as in Nascimento (2003), Maharaj (2010), Mutlu and Aydin (2013), Brandemberg & Messias (2016), Messias (2018), Messias and Brandemberg (2021). In what concerns to the other items of Task 1, the participants evoked similar elements to justify their responses. In chart 2, we present some of the participants responses[3].

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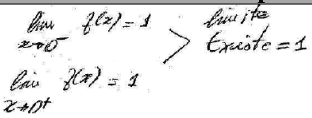
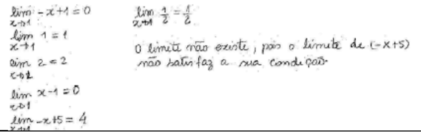
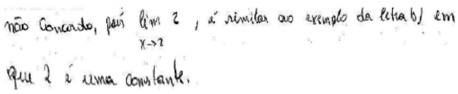
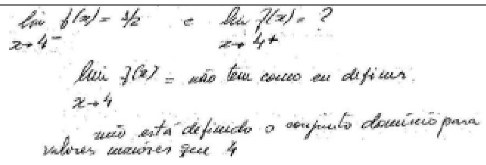
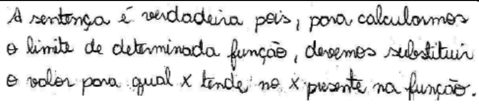
[2] We analyzed the responses given by five students, which were identified as S1, S2, S3, S4 and S5.
[3] Participants' original answers were in Portuguese. In this paper, the principal ideas of their responses were translated to English.

Chart 2: Multiple Responses given to task

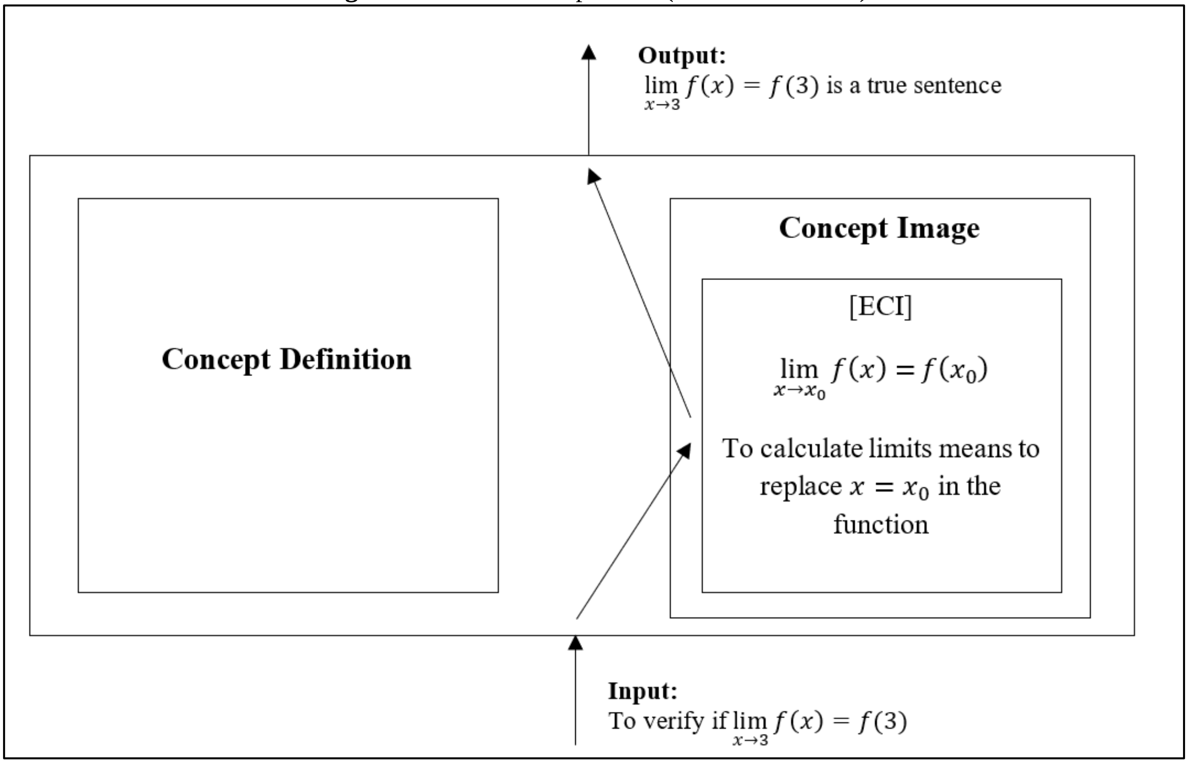
Student	Item	Response/Explanation	
S2	1.1		Student affirms that the "limit exists" based on the incoherent graphic he drew.
S3	1.2		Student affirms that "the limit does not exist because $-x+5$ does not satisfy its condition".
S1	1.3		Student 1 disagrees that L does not exist, since, according to him, $f(x)=2$ is a constant function, which, for him, makes it possible for the limit to exist.
S4	1.4		Student affirms that there is no way to define because there are no values for $x>4$. Student affirms that $\lim_{x\to 4^-} f(x) = 1/2$, which, in fact, is the value for $f(4)$.
S5	1.5		Student affirms that $\lim_{x\to 3} f(x) = f(3)$, since "to calculate certain limit, we should replace the value in which x tends in the x of the function"

Source: The author.

Students' evoked concept images brought multiple elements that probably led them to certain misconceptions. As mentioned before, we believe that Concept Image and Concept Definition Schemes may be powerful tools to analyze and conjecture about (possible) mistakes, potential (or cognitive) conflict factors, difficulties in the learning of any object etc. It is presented some CI and CD Schemes related to what was conjectured about students' comprehension of limit and/or continuity of a function (see figures below)[4].

[4] In all the schemes, the Evoked Concept Image is presented as a subset of the Concept Image, once it does not necessarily represent everything an individual knows about an object, but what was mobilized at a certain mathematical situation.

Figure 6 - Intuitive Response 1 (Task 1 – item 1.5)



Source: The author.

The intuitive response presented in figure 6 is referred S5's answer to item 1.5 (see chart 2). Such Evoked Concept Image might be related to conflicts about what $\lim_{x\to x_0} f(x)$ and $f(x_0)$ means. It is possible that his whole comprehension about continuity is attached to the condition $\lim_{x\to x_0} f(x) = f(x_0)$, or that replacing $x = x_0$ in the function is the only way to calculate limits. This can be either seen as a potential conflict factor or a result of previous learning experiences that strongly explored situations, in which such operational practice was enough to calculate the limit (e.g., polynomial functions).

Student S3 responses were also intuitive since the subject did not mobilize elements of Formal Concept Definition to justify the limit's (non) existence. It was also evident that lateral limits were not evoked (see figure 7).

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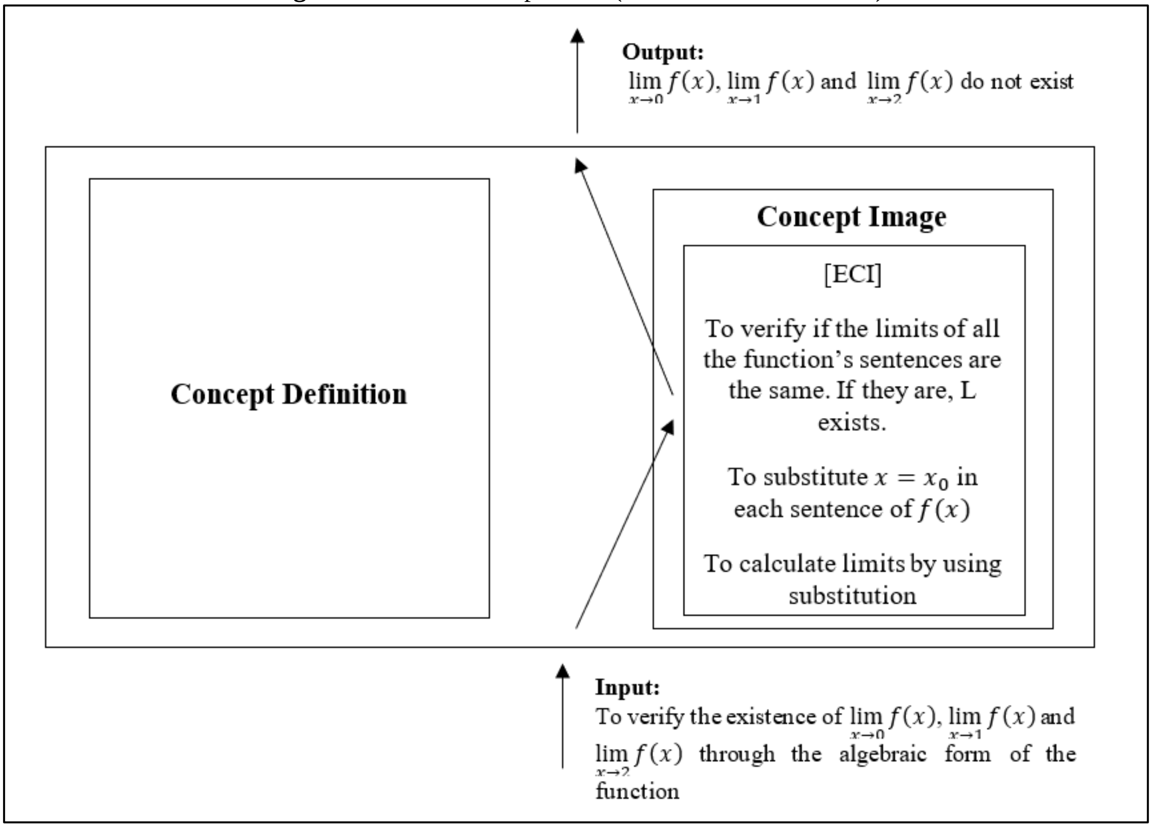
Figure 7 - Intuitive Response 1 (Task 1 – item 1.1 to 1.3)

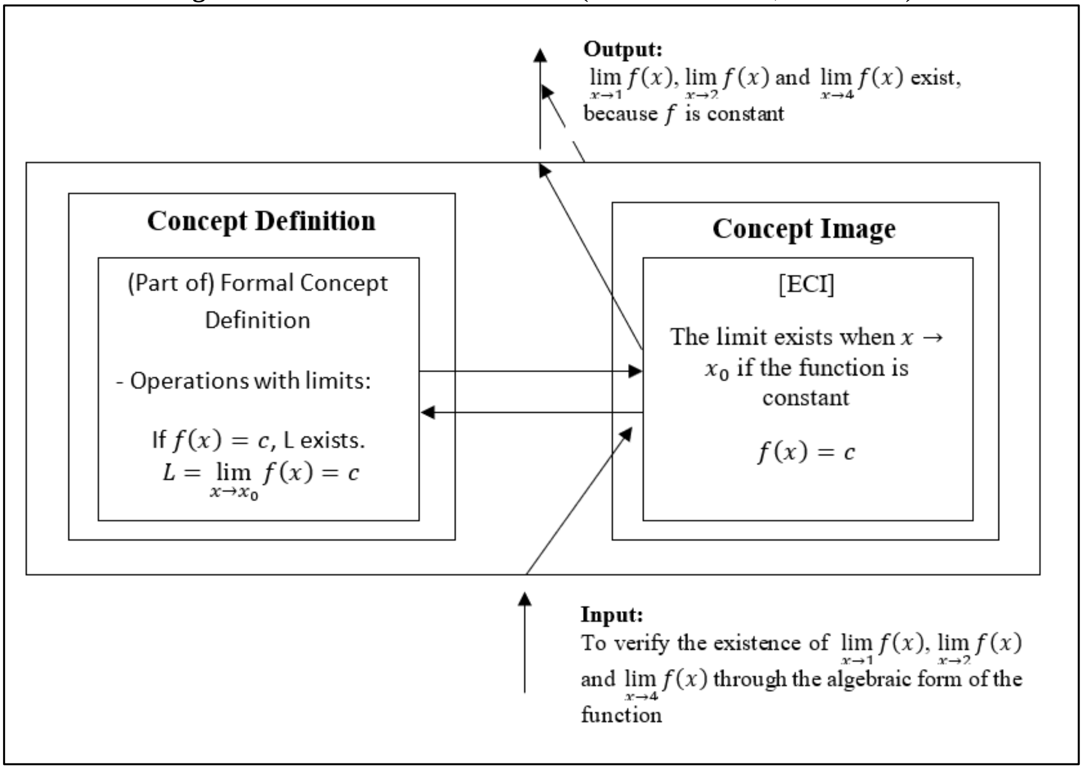


Source: The author.

The intuitive response presented in figure 7 refers to S3's responses, including the one shown in chart 2. It was observed that to calculate the limit of a function written in multiple sentences was, for many of the participants, a cognitive conflict factor.

In figure 8, we present a Partial-Intuitive Deduction to conjecture about S1's response to items 1.2, 1.3 and 1.4 (see chart 2). It is believed that it looks like the one characterized by Vinner (1991) as Deduction following intuitive thought. The difference is that the individual seems to consult the Concept Definition Cell, but then, s/he goes back to the concept image cell to respond to what was asked. Such behavior can (or cannot) be problematic. It depends on if his/her CI has any misconception related to one or more objects involved in the task.

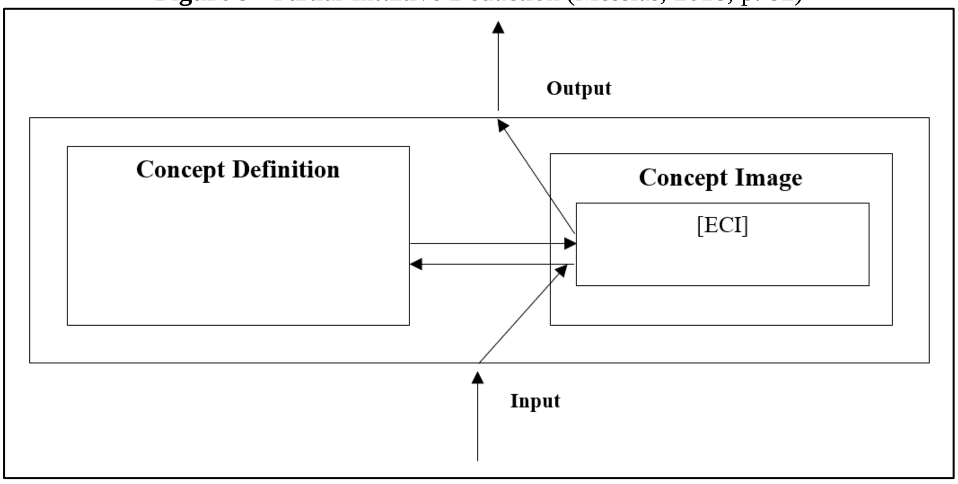
Figure 8 - Partial-Intuitive Deduction (Task 1- items 1.2, 1.3 and 1.4)



Source: The author.

The partial-intuitive deduction is generalized by Messias (2018) as follows:

Figure 9 - Partial-Intuitive Deduction (Messias, 2018, p. 82)



Source: The author.

In task 2, students' answers allowed us to verify their understanding of the concept of limit through the evoked concept image in their explanations. In chart 3, we present the participants' responses.

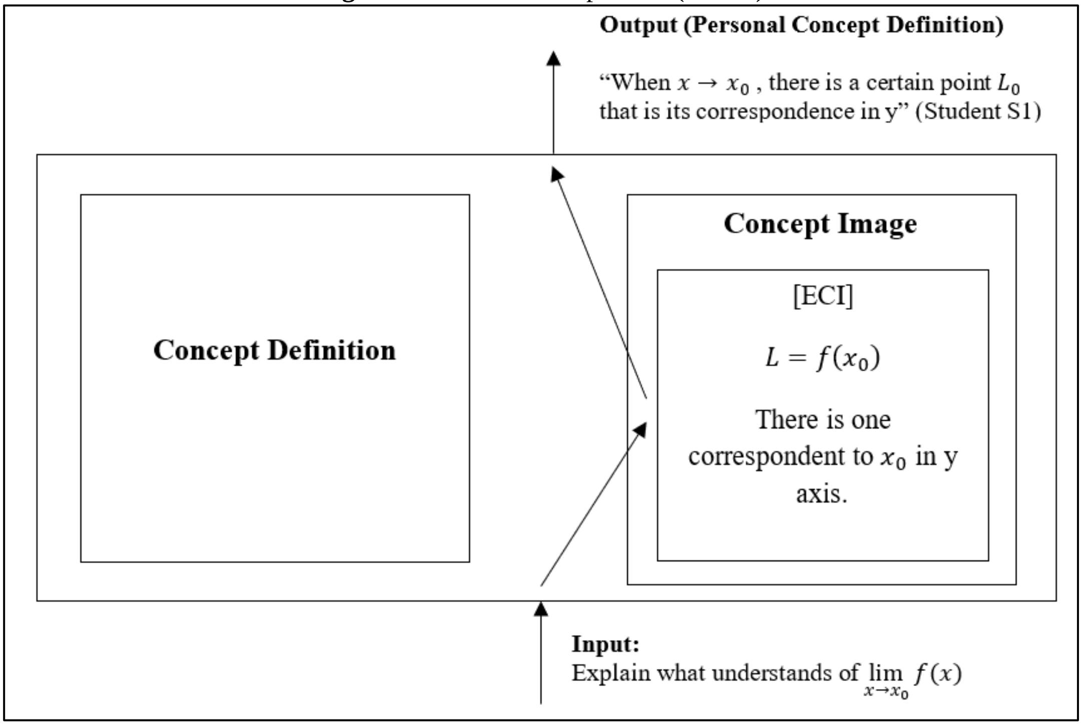
Chart 3: Personal Concept Definitions of Limit.

Student	Response/Explanation	
S1	quando $x \to x_0$, tem um certo ponto L_0 que é seu correspondente em y.	Student connected L to a functional relation, established between the axes.
S2	Qualquer valor que você colocar no intervalo esquerdo, e também direito, tenderá para o mesmo limite, no caso L. Então existe um limite para x quando $x \to x_0$ e é L.	Student evoked the idea of approaching from both sides of a certain x_0 which leads to corresponding values in y axis.
S3	o $\lim_{x \to x_0} f(x) = L$, é quando o limite da função f(x), onde x tende a x_0 é igual ao um determinado valor, como L.	Student wrote a “translation” to the notation $\lim_{x \to x_0} f(x)$; no other elements were mobilized.
S4	Entendo como uma aproximação de $x_0 \in D(f)$ pelos extremos -x e +x, que pela relação funcional provoca também uma aproximação de extremos -y e +y do ponto L que pode estar ou não no contradomínio (ou imagem devido a incerteza) na função.	Student connected L to a functional relation, established between the axes; confused about what image and range mean.
S5	Para tal raciocínio, podemos considerar uma reta cujo comprimento é L. Assim, para calcular o limite da função representada pela reta, devemos partir de um valor final (x) para um valor inicial (x_0).	L can be represented as a line and its length is calculated by the difference between initial and final position.

Source: The author.

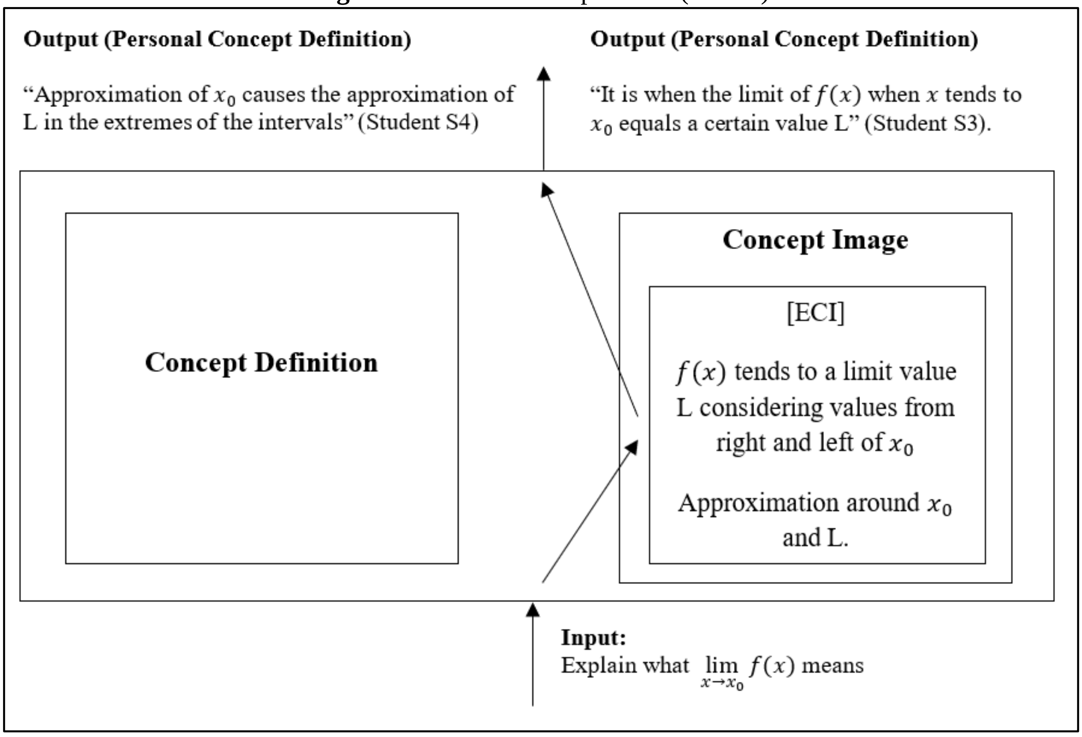
Considering students’ responses, it was noticed that they attached the limit to the functional relation established between the axes, which may lead to conceptual confusions about what $\lim_{x \to x_0} f(x)$ and $f(x_0)$ mean. It was also observed that their concept images brought reference to lateral limits and a dynamic conception of movement, as well as seen in Tall e Vinner (1981), Cornu (1983), Cornu (1991), Cottril et al. (1996), Przenioslo (2004), Sarvestani (2011), Swinyard (2011), Amatangelo (2013), Denbel (2014), Oh (2014). Only S5’s ECI had elements that were not expected, considering the teaching experience in Calculus and/or what has been discussed about the (difficulties in) learning of limits by the literature. According to him, L represents a line, and its length can be obtained from the difference between final and initial positions. In a general way, students’ responses were intuitive, and they are represented by figures 10, 11 and 12.

Figure 10 - Intuitive Response 1 (Task 2)



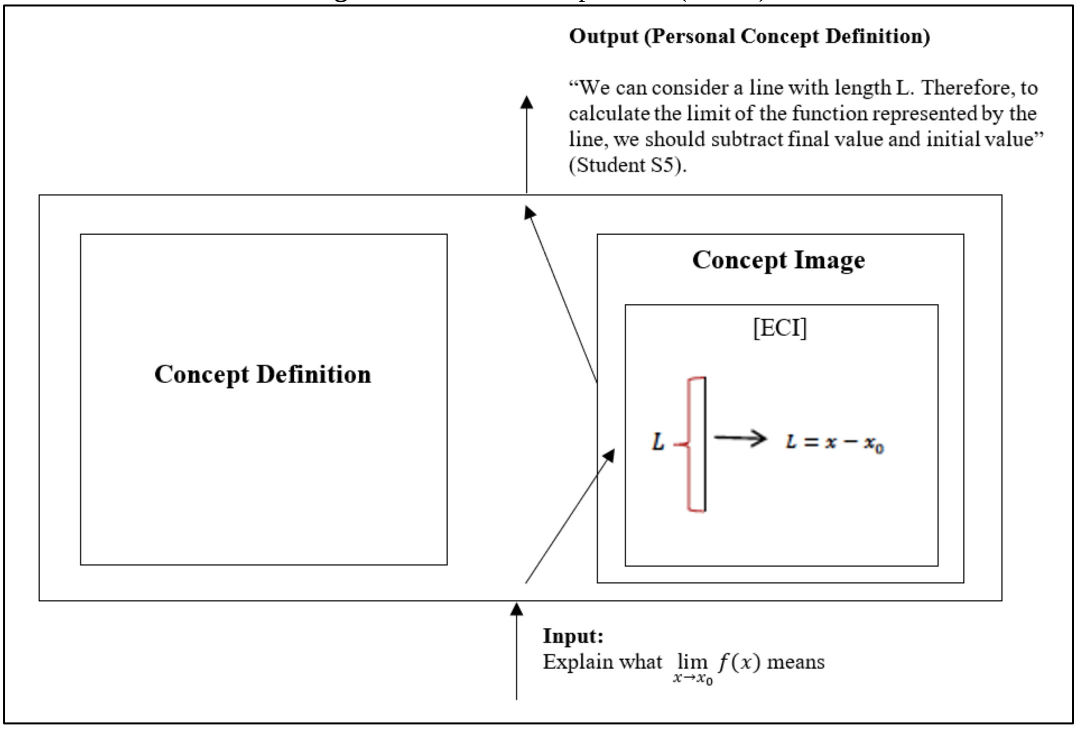
Source: The author.

Figure 11 - Intuitive Responses 2 (Task 2)



Source: The author.

Figure 12 - Intuitive Responses 3 (Task 2)



Source: The author.

In task 3, students related the asymptotes to the conditions of the domain and range of the function, besides mobilizing the dynamic conception of movement to f (according to them, asymptotes restrict the function's movement). One of them interpreted the asymptotes as boarders that do not let certain values to be attained by the function. Only one student evoked elements from the formal definition of both, horizontal and vertical asymptotes (S4). In his explanations, he mobilized the idea of limits involving infinity. All of students' responses can be seen in chart 3.

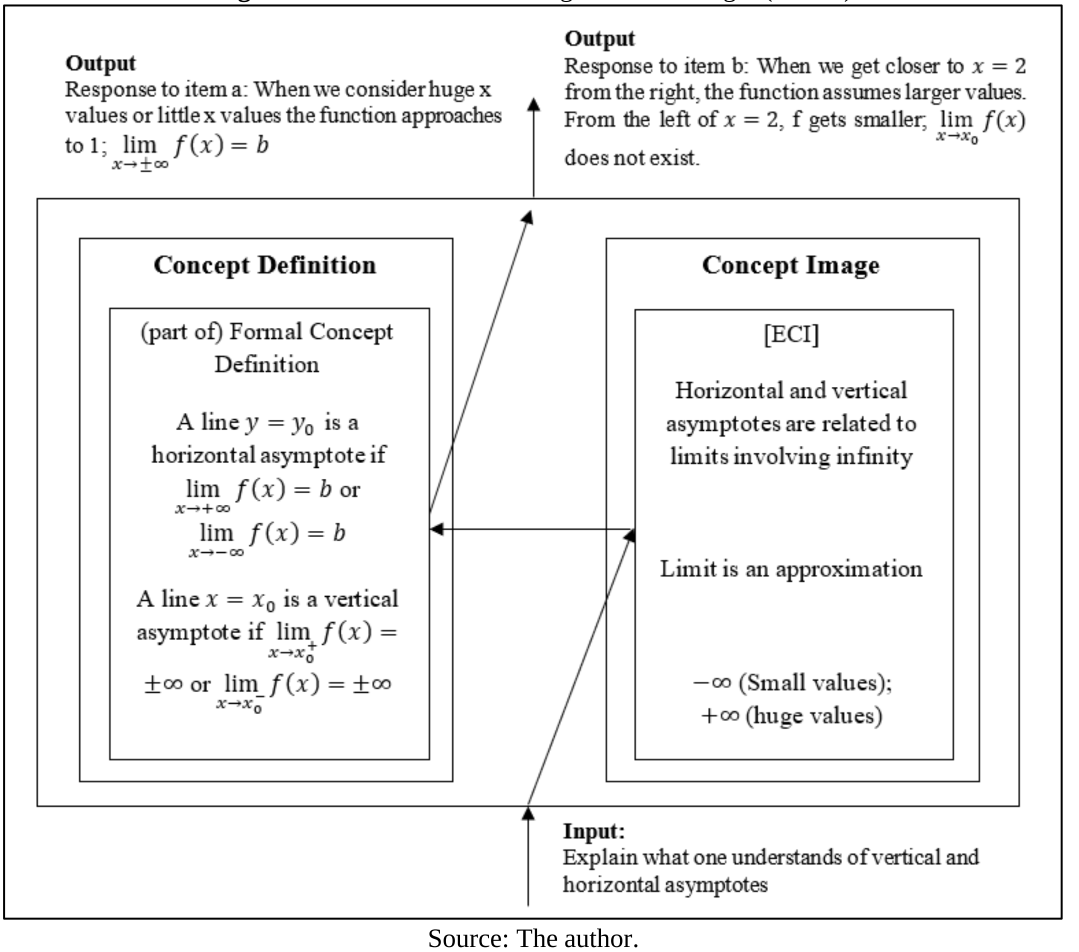
Chart 4 - Responses to task 3

Student	Responses in Portuguese (Item a and b)	
S1	Seria uma espécie de limite para a função, onde ela nunca terá valores menores que 1 em Y	Um limite onde a função não apresentará valores maiores que −2 em X.
S2	$1 = \frac{x+3}{x+2}$ $x+2 = x+3$ $x - x = 3-2$ $0x = 1$ não existe definição do conjunto imagem p/ y = 1, ou seja, $Im = \{y \in \mathbb{R} / y \neq 1\}$.	$f(-2) = \frac{-2+3}{-2+2=0}$ não é condição de ~~indeterminação~~ indefinição, logo não existe valores definidos para $x = -2$; pela esquerda tende a $-\infty$ e pela direita a $+\infty$. $Dom = \{x \in \mathbb{R} / x \neq -2\}$.
S3	Ela explica onde o gráfico da função mais se aproxima e também que o valor y = 1 não está no contradomínio	A assíntota ela, aqui também, explica onde o gráfico da função mais se aproxima, e também explica que o valor de x = 1 não está no domínio
S4	Que o $\lim_{x \to \infty} f(x) = 1$. Em escrito: que na medida que tomarmos valores muito grandes ou muito pequenos (negativos) a função se aproxima de 1	Que o $\lim_{x \to -2} f(x) = \nexists$ mas que a função se aproxima de valores muito grandes (+∞), quando x se aproxima pela direita de −2 e que ao mesmo tempo se aproxima de valores muito pequenos (−∞), quando x se aproxima pela esquerda de −2.
S5	Significa que o eixo das abscissas se deslocou 1 unidade para cima determinando assim um "novo" e restringido movimento ao gráfico	Significa que o eixo das ordenadas se deslocou 2 unidades para a esquerda determinando assim um "novo" e restringido movimento ao gráfico

Source: The author.

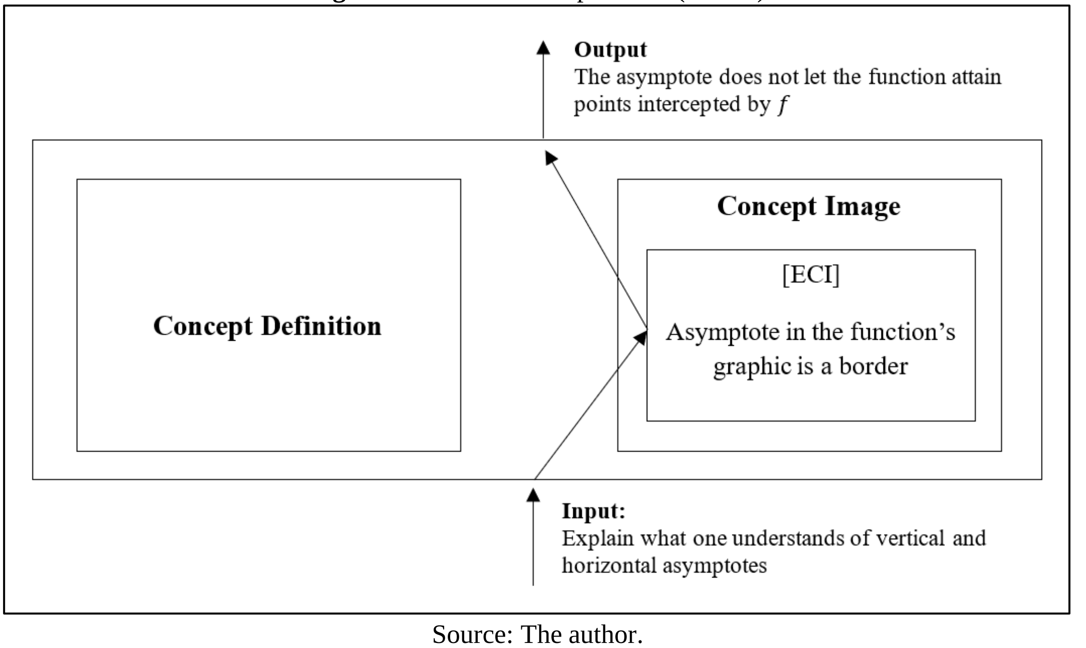
Students' responses to task 3 were all intuitive, except for S4's, whose answers seemed to be based on a deduction following intuitive thought (see figure 13). The intuitive responses brought elements that characterized asymptotes as borders that avoid certain values to be attained, restrict the function's movement, and exclude elements from its domain and range (see figures 14 and 15).

Figure 13 - Deduction following intuitive thought (Task 3)



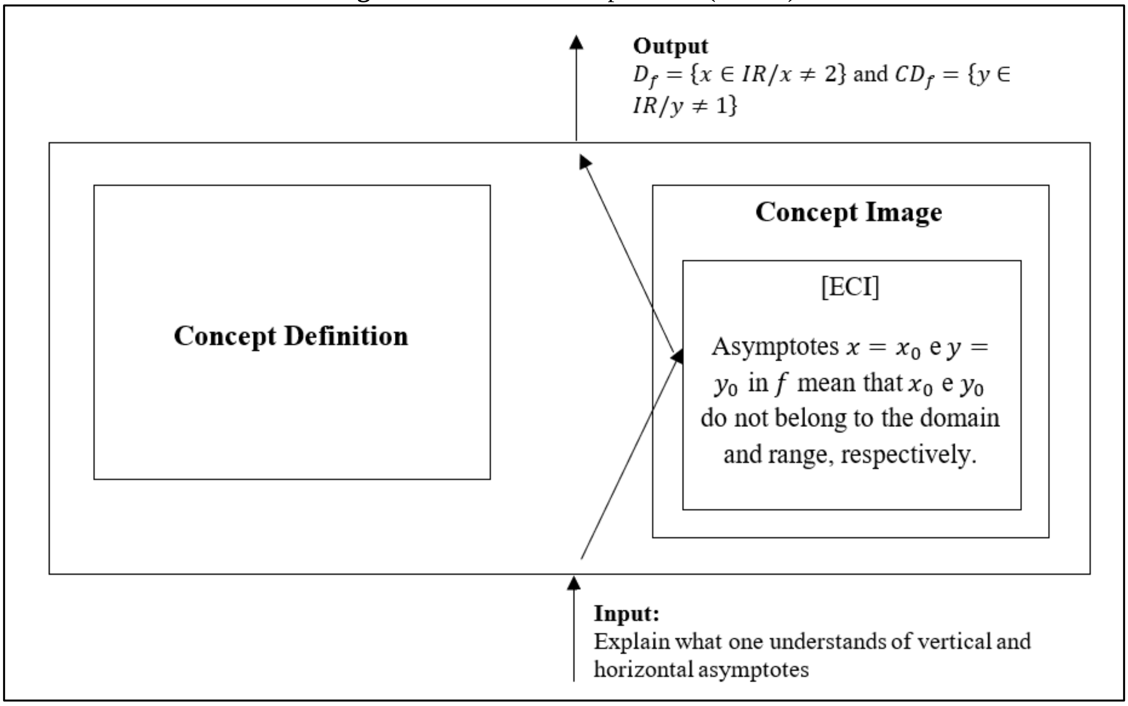
Source: The author.

Figure 14 - Intuitive Responses 1 (Task 3)



Source: The author.

Figure 15 - Intuitive Responses 2 (Task 3)



Source: The author.

In task 4, most of the students gave intuitive responses to what was asked. Student S1, for example, mobilized the intervals $(x_0 - \delta, x_0 + \delta)$ to emphasize the approximation around x_0, as observed in some of the expressions that were used to describe his personal concept definition (e.g., will tend, to the right of, to the left of). Besides, both, S1 and S3, evoked comprehensions related to limit of a function at a point, not infinite limits.

The infinite limits were mobilized in students S2, S4 and S5 responses. We observed that S2 evoked lateral limits (around x_0) and bilateral limits. S4 evoked the idea of approximation around a point (in this case, x_0), besides the comprehension of (de)creasing the f values, which the student assigned to his personal concept definition of infinite limits. Student S5 used the terms "positive infinite" and "negative infinite" to explain that the function was away from the origin, tending, respectively, to $+\infty$ and $-\infty$. Below, in chart 5, we present some of the students' reponses to task 4.

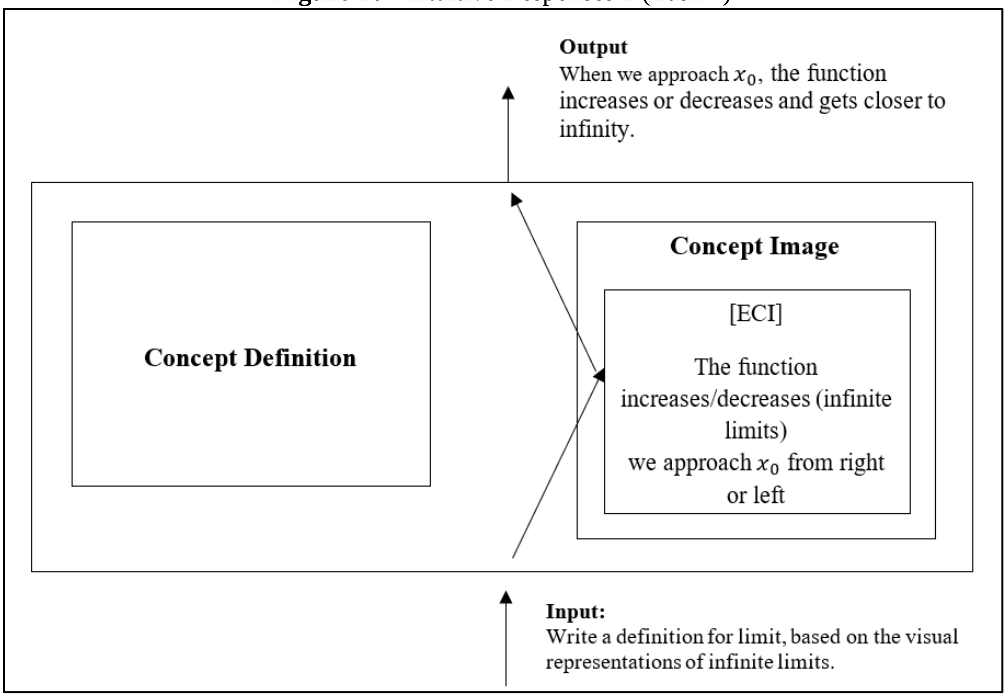
Chart 5 - Responses to task 4

Student	Responses in Portuguese (Item a and b)	
S2	$\lim_{x\to x_0^-} f(x) = +\infty$ e $\lim_{x\to x_0^+} f(x) = +\infty$ logo, $\lim_{x\to x_0} f(x) = +\infty$	$\lim_{x\to x_0^-} f(x) = -\infty$ e $\lim_{x\to x_0^+} f(x) = -\infty$ logo, $\lim_{x\to x_0} f(x) = -\infty$.
S3	O limite da função $f(x)$, onde x tende a x_0, tem como o limite igual a B. $\lim_{x\to x_0} f(x) = B$	O limite da função $f(x)$, onde x tende a x_0, tem como limite igual a $-B$. $\lim_{x\to x_0} f(x) = -B$.
S4	Que o limite de $f(x)$, quando x se aproxima de x_0 pela direita e pela esquerda Cresce sem limites (que chamamos amigavelmente de $+\infty$)	Que o limite de $f(x)$ quando x se aproxima de x_0 pela direita e esquerda decresce sem limite (onde chamamos de $-\infty$)

Source: The author.

Students' responses were intuitive and mostly based on dynamic perceptions of the limit concept, as shown in figures16.

Figure 16 - Intuitive Responses 1 (Task 4)



Source: The author.

In task 5, we noticed that none of the participants considered that the existence of the limits is necessarily related to the absence of indeterminations but highlighted the importance of "math mechanisms" and "algebraic manipulations" to "take it off" and then, find the limit at a point, as exemplified in chart 6.

Chart 6 - Responses to task 5

Student	Response/Explanation	
S1	Este resultado significa uma indeterminação, onde não dá para afirmar que f(x) tem limite sem uma manipulação algébrica Ex: $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$	Indetermination means there is limit only after algebraic manipulation
S4	Provavelmente e uma função com um "buraco", um ponto não definido na sua lei de formação, ainda não investiguei, mas é a minha melhor hipótese, já que eles usam para casos simples de quociente de funções algébricas cujo o domínio do denominador não abrange o zero da função. Ex: $\lim_{x \to -1} \frac{x^2 + x + 1}{x + 1}$	Indetermination indicates the existence of a hole, a point that is not defined in the function's domain; it happens with rational functions
S5	O resultado determinado pelo estudante se trata de uma indeterminação. Isso significa que o limite da função não será reconhecido se não for utilizado um mecanismo matemático que saia da indeterminação. Ex: $\lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \frac{2 - 2}{2^2 - 4} = \frac{0}{0}$ (indeterminação)	The limit will only exist if one uses mathematical mechanisms to eliminate the indetermination

Source: The author.

The examples presented by the students in their explanations were similar, once they represented limits of rational functions such as $\lim_{n\to\infty} \frac{x^2-a^2}{x-a}$ or $\lim_{n\to\infty} \frac{x-a}{x^2-a^2}$, which are common in Calculus books. S4 wrote $\lim_{x\to x_0} f(x) = \frac{1}{0}$. Despite the fact, he did not mention anything about infinite limits; he\she reinforced the idea of emergency of null denominator as a practice of calculating limits. Figures 17 illustrates possible elements that were evoked in their responses, which we believe were intuitive.

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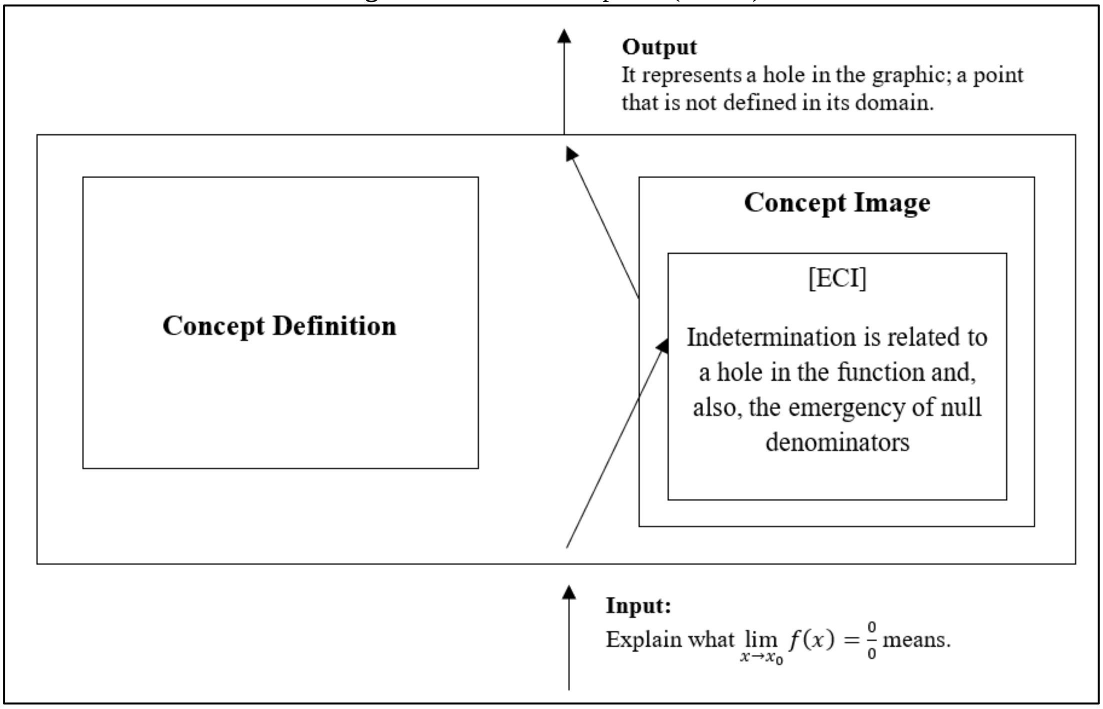
Figure 17: Intuitive Response (Task 5)



Source: The author.

The student's responses allowed us to conjecture about the elements that were part of their concept image and that were evoked during each of the tasks. The plurality of interpretations pointed in other research, the participants' evocations in this study, and our teaching experience in Calculus permitted us to attain the main goal of this paper, which was to analyze one's understanding of the concept of limit of a function based on Concept Image and Concept Definition schemes.

4 CONCLUSIONS

The CI and CD Schemes presented in this paper permitted us to conjecture about students' understanding of the concept of limit (and other adjacent concepts), which were related to multiple comprehensions evoked in their responses, such as the ones listed below:

[C1] – Limit's existence depends on the lateral limits.

[C2] – Limit is an approximation around x_0.

[C3] – Infinite limits: once we get closer to x_0, the function increases or decreases infinitely.

[C4] – To calculate the limit of a function written in more than one sentence, one should substitute $x = x_0$ in all the sentences that compose $f(x)$.

[C5] - $\lim_{x \to x_0} f(x)$ means that when $x \to x_0, f(x) \to L$.

[C6] - $\lim_{x \to x_0} f(x)$ is always the same of $f(x_0)$.

[C7] – L is the length of a line.

[C8] – Indetermination implies in a hole in the graphic of the function and, consequently, in the emergence of null denominators.

[C9] – An asymptote represents a border that prevent

[C10] – The presence of an asymptote implies that the calculus of the limit needs to be done through approximation.

[C11] – If $x = x_0$ is a vertical asymptote, then x_0 does not belong to the domain; if $y = y_0$ is a horizontal asymptote, thus y_0 does not belong to the range.

[C12] – An asymptote restricts the movement of the function.

[C13] – An asymptote (horizontal or vertical) implies in the emergency of infinite limits.

It is important to emphasize, once again, that when an individual solves a cognitive task, part of his/her concept image activates. It is possible that all his/her knowledge related to a certain object is represented by the evoked concept image or, depending on what is asked in a different task, he/she can mobilize other comprehensions. That means that what was listed from [C1] to [C13] are not necessarily everything they know about limits, and it does not mean that they were evoked simultaneously.

Finally, we reinforce that the results obtained in this study were relevant, since it allowed us to verify possible conflicts that are part of students' concept images about limits. These results may lead to future studies, through which may be possible to develop didactic proposals to be applied to enhance one's understanding in Calculus.

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